TD1 – Intégration par parties – Proposition de correction

Calculer les intégrales suivantes (par parties)

I = $\int_{-3}^{0}(x+1)\,e^{-x}\,dx$

On pose $\begin{cases} u(x) = x+1 \\ v'(x) = e^{-x} \end{cases}$ on obtient : $\begin{cases} u'(x) = 1 \\ v(x) = -e^{-x} \end{cases}$

I = $[\,(x+1).(-e^{-x})\,]_{-3}^{0} - \int_{-3}^{0} 1.(-e^{-x})dx$

I = $[\,(x+1).(-e^{-x})\,]_{-3}^{0} - [\,e^{-x}\,]_{-3}^{0}$

I = $[\,-(x+1).e^{-x}\,]_{-3}^{0} - [\,e^{-x}\,]_{-3}^{0}$

I = $[\,(-x-1)-1).e^{-x}\,]_{-3}^{0}$

I = $[\,(-x-2).e^{-x}\,]_{-3}^{0}$

I = $(-0-2).e^{0} - (3-2).e^{3}$

Finalement $\boxed{I = -2 - e^{3}}$

J = $\int_{0}^{2}(-2x+4)\,e^{x}\,dx$

On pose $\begin{cases} u(x) = -2x+4 \\ v'(x) = e^{x} \end{cases}$ on obtient : $\begin{cases} u'(x) = -2 \\ v(x) = e^{x} \end{cases}$

J = $[\,(-2x+4).\,e^{x}\,]_{0}^{2} - \int_{0}^{2} -2e^{x}dx$

J = $[\,(-2x+4).\,e^{x}\,]_{0}^{2} + 2\int_{0}^{2} e^{x}dx$

J = $[\,(-2x+4).\,e^{x}\,]_{0}^{2} + 2\,[\,e^{x}\,]_{0}^{2}$

J = $[\,(-2x+6).\,e^{x}\,]_{0}^{2}$

J = $(-2.2+6).\,e^{2} - (-2.0+6).\,e^{0}$

Finalement $\boxed{J = 2e^{2} - 6}$

K = $\int_0^1 \ln(x+3)dx$

On pose $\begin{cases} u'(x) = 1 \\ v(x) = \ln(x+3) \end{cases}$ on obtient : $\begin{cases} u(x) = x \\ v'(x) = \dfrac{1}{x+3} \end{cases}$

K = $[\, x \ln(x+3)]_0^1 - \int_{-3}^{0} \dfrac{x}{x+3}dx$

K = $[\, x \ln(x+3)]_0^1 - \int_{-3}^{0} \dfrac{x+3-3}{x+3}dx$

K = $[\, x \ln(x+3)]_0^1 - \int_{-3}^{0} \dfrac{x+3}{x+3} - \dfrac{3}{x+3}dx$

K = $[\, x \ln(x+3)]_0^1 - \int_{-3}^{0} 1 - 3.\dfrac{1}{x+3}dx$

K = $[\, x \ln(x+3)]_0^1 - [\, x - 3\ln(x+3)]_0^1$

K = $[(x-3).\ln(x+3) - x\,]_0^1$

K = $(1-3).\ln(1+3) - 1 - (0-3).\ln(0+3) + 0$

Finalement $\boxed{K = -2\ln 4 + 3\ln 3}$

L = $\int_1^e x^2 \ln x\, dx$

On pose $\begin{cases} u'(x) = x^2 \\ v(x) = \ln x \end{cases}$ on obtient : $\begin{cases} u(x) = \dfrac{x^3}{3} \\ v'(x) = \dfrac{1}{x} \end{cases}$

L = $[\dfrac{x^3}{3}\ln x\,]_1^e - \int_1^e \dfrac{x^3}{3x}dx$

L = $[\dfrac{x^3}{3}\ln x\,]_1^e - \dfrac{1}{3}\int_{-3}^{0} x^2 dx$

L = $[\dfrac{x^3}{3}\ln x\,]_1^e - \dfrac{1}{3}[\,\dfrac{x^3}{3}\,]_1^e$

L = $[\dfrac{x^3}{3}(\ln x - \dfrac{1}{3})]_1^e$

L = $\dfrac{e^3}{3}(\ln e - \dfrac{1}{3}) - (\dfrac{1^3}{3}(\ln 1 - \dfrac{1}{3}))$

$L = \frac{e^3}{3}(1 - \frac{1}{3}) - (\frac{1}{3}(-\frac{1}{3}))$

$L = \frac{e^3}{3}(\frac{2}{3}) - (-\frac{1}{9})$

Finalement $\boxed{L = \frac{2e^3+1}{9}}$

$M = \int_1^3 t \ln t \, dt$

On pose $\begin{cases} u'(t) = t \\ v(t) = \ln t \end{cases}$ on obtient : $\begin{cases} u(t) = \frac{t^2}{2} \\ v'(t) = \frac{1}{t} \end{cases}$

$M = [\frac{t^2}{2}\ln t]_1^3 - \int_1^3 \frac{t^2}{2t}dt$

$M = [\frac{t^2}{2}\ln t]_1^3 - \int_1^3 \frac{t}{2}dt$

$M = [\frac{t^2}{2}\ln t]_1^3 - \frac{1}{2}[\frac{t^2}{2}]_1^3$

$M = [\frac{t^2}{2}(\ln t - \frac{1}{2})]_1^3$

$M = \frac{9}{2}(\ln 3 - \frac{1}{2}) - \frac{1}{2}(\ln 1 - \frac{1}{2})$

$M = \frac{9}{2}\ln 3 - \frac{9}{4} + \frac{1}{4}$ Finalement $\boxed{M = \frac{9}{2}\ln 3 - 2}$

$N = \int_0^{\frac{\pi}{2}} t \cos t \, dt$

On pose $\begin{cases} u(t) = t \\ v'(t) = \cos t \end{cases}$ On obtient : $\begin{cases} u'(t) = 1 \\ v(t) = \sin t \end{cases}$

$N = [t \sin t]_0^{\frac{p}{2}} - \int_0^{\frac{p}{2}} \sin t dt$

$N = [t \sin t]_0^{\frac{p}{2}} + \int_0^{\frac{p}{2}} -\sin t dt$

$N = [t \sin t]_0^{\frac{p}{2}} + [\cos t]_0^{\frac{p}{2}}$

$N = (\frac{p}{2}\sin\frac{p}{2} - 0.\sin 0) + (\cos \frac{p}{2} - \cos 0)$

Finalement $\boxed{N = \frac{p}{2} - 1}$

$O = \int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} e^{x} \cos x \, dx$ On effectuera deux intégrations par parties successives.

On pose $\begin{cases} u(x) = e^{x} \\ v'(x) = \cos x \end{cases}$ on obtient : $\begin{cases} u'(x) = e^{x} \\ v(x) = \sin x \end{cases}$

$O = [e^{x} \sin x]_{-\frac{p}{2}}^{\frac{p}{2}} - \int_{-\frac{p}{2}}^{\frac{p}{2}} e^{x} \sin x dx$

On pose $\begin{cases} u(x) = e^{x} \\ v'(x) = \sin x \end{cases}$ on obtient : $\begin{cases} u'(x) = e^{x} \\ v(x) = -\cos x \end{cases}$

$O = [e^{x} \sin x]_{-\frac{p}{2}}^{\frac{p}{2}} - \left([e^{x}(-\cos x)]_{-\frac{p}{2}}^{\frac{p}{2}} - \int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} e^{x}(-\cos x)dx \right)$

$O = [e^{x} \sin x]_{-\frac{p}{2}}^{\frac{p}{2}} - [e^{x}(-\cos x)]_{0}^{\frac{p}{2}} - \int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} e^{x} \cos x \, dx$

$O = [e^{x} \sin x]_{-\frac{p}{2}}^{\frac{p}{2}} - [e^{x}(-\cos x)]_{0}^{\frac{p}{2}} - O$

$2O = [e^{x} \sin x]_{-\frac{p}{2}}^{\frac{p}{2}} + [e^{x} \cos x]_{-\frac{p}{2}}^{\frac{p}{2}}$

$2O = [e^{x}(\sin x + \cos x)]_{-\frac{p}{2}}^{\frac{p}{2}}$

$2O = e^{\frac{p}{2}}(\sin \frac{p}{2} + \cos \frac{p}{2}) - e^{-\frac{p}{2}}(\sin(-\frac{p}{2}) + \cos(-\frac{p}{2}))$

$2O = e^{\frac{p}{2}}(1+0) - e^{-\frac{p}{2}}(-1+0)$

$2O = e^{\frac{p}{2}} + e^{-\frac{p}{2}}$ Finalement $\boxed{O = \frac{e^{\frac{p}{2}} + e^{-\frac{p}{2}}}{2}}$

Rq : $O = \text{ch} \frac{p}{2}$ (voir formulaire → cosinus hyperbolique)

$P = \int_{0}^{\Pi} e^{-x} \sin 4x dx$. On effectuera deux intégrations par parties successives.

On pose $\begin{cases} u(x) = e^{-x} \\ v'(x) = \sin 4x \end{cases}$ on obtient : $\begin{cases} u'(x) = -e^{-x} \\ v(x) = -\frac{1}{4}\cos 4x \end{cases}$

$$P = \left[e^{-x}\left(-\frac{1}{4}\right)\cos 4x \right]_0^{\pi} - \int_0^{\pi} -e^{-x}\left(-\frac{1}{4}\right)\cos 4x\,dx$$

On pose $\begin{cases} u(x) = -e^{-x} \\ v'(x) = -\frac{1}{4}\cos 4x \end{cases}$ on obtient : $\begin{cases} u'(x) = e^{-x} \\ v(x) = -\frac{1}{16}\sin 4x \end{cases}$

$$P = \left[e^{-x}\left(-\frac{1}{4}\right)\cos 4x \right]_0^{\pi} - \left(\left[-e^{-x}\left(-\frac{1}{16}\right)\sin 4x \right]_0^{\pi} - \int_0^{\pi} e^{-x}\left(-\frac{1}{16}\right)\sin 4x\,dx \right)$$

$$P = \left[e^{-x}\left(-\frac{1}{4}\right)\cos 4x \right]_0^{\pi} - \left[\frac{1}{16} e^{-x}\sin 4x \right]_0^{\pi} - \frac{1}{16}\int_0^{\pi} e^{-x}\sin 4x\,dx$$

$$P = \left[e^{-x}\left(-\frac{1}{4}\right)\cos 4x \right]_0^{\pi} - \left[\frac{1}{16} e^{-x}\sin 4x \right]_0^{\pi} - \frac{1}{16} P$$

$$P + \frac{1}{16} P = \left[-e^{-x}\left(\frac{1}{4}\cos 4x + \frac{1}{16}\sin 4x\right) \right]_0^{\pi}$$

$$\frac{17}{16} P = -e^{-\pi}\left(\frac{1}{4}\cos 4\pi + \frac{1}{16}\sin 4\pi\right) - (-e^{0})\left(\frac{1}{4}\cos 0 + \frac{1}{16}\sin 0\right)$$

$$\frac{17}{16} P = -e^{-\pi}\left(\frac{1}{4} + 0\right) + \left(\frac{1}{4} + 0\right)$$

$$\frac{17}{16} P = \frac{1 - e^{-\pi}}{4}$$

$$P = \frac{16(1 - e^{-\pi})}{4.17}$$

Finalement $$\boxed{P = \frac{4(1 - e^{-\pi})}{17}}$$

TD2 – Intégration par changement de variable – Proposition de correction

Exercice 1

I = $\displaystyle\int_{-1}^{1}\frac{dx}{\sqrt{3x+5}}$ On posera u= 3x + 5 …

On pose u = 3x+5 soit 1.du = 3.dx donc dx = $\frac{du}{3}$

$$I = \int_{3(-1)+5}^{3.1+5}\frac{du}{3\sqrt{u}}$$

$$= \int_{2}^{8}\frac{2du}{3.2\sqrt{u}}$$

$$= \frac{2}{3}\int_{2}^{8}\frac{du}{2\sqrt{u}}$$

$$= \frac{2}{3}[\sqrt{u}]_{2}^{8}$$

$$= \frac{2}{3}(\sqrt{8}-\sqrt{2})$$

$$= \frac{2}{3}(2\sqrt{2}-\sqrt{2})$$

Finalement $\boxed{I = \frac{2\sqrt{2}}{3}}$

J = $\displaystyle\int_{0}^{1}(x+1)e^{x^2+2x}dx$ On posera u = x ² + 2 x …

On pose u = x²+2x soit du =(2x+2) dx donc du = 2(x+1) dx

$$J = \int_{0^2+2.0}^{1^2+2.1}e^{u}\frac{du}{2}$$

$$= \frac{1}{2}\int_{0}^{3}e^{u}du$$

$$= \frac{1}{2}[e^{x}]_{0}^{3}$$

$$= \frac{1}{2}(e^{3}-e^{0})$$

Finalement $\boxed{J = \frac{e^3-1}{2}}$

K = $\int_{-1}^{0} t\ \sqrt{t+2}\ dt$ On posera u = t + 2

On pose u = t + 2 soit t = u– 2 et du = dt

$$K = \int_{-1+2}^{0+2} (u-2)\sqrt{u}\ du$$

$$= \int_{1}^{2} (u-2)u^{\frac{1}{2}}\ du$$

$$= \int_{1}^{2} u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\ du$$

$$= \left[\ \frac{1}{\frac{3}{2}+1} u^{\frac{3}{2}+1}\right]_1^2 - 2\left[\frac{1}{\frac{1}{2}+1} u^{\frac{1}{2}+1}\right]_1^2$$

$$= \left[\ \frac{2}{5} u^{\frac{5}{2}}\right]_1^2 - 2\left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^2$$

$$= \frac{2}{5}.2^{\frac{5}{2}} - \frac{2}{5}.1^{\frac{5}{2}} - 2\ \left(\frac{2}{3}.2^{\frac{3}{2}} - \frac{2}{3}.1^{\frac{3}{2}}\right)$$

$$= \frac{2}{5}.\sqrt{2^5} - \frac{2}{5} - 2\ \left(\frac{2}{3}.\sqrt{2^3} - \frac{2}{3}\right)$$

$$= \frac{2}{5}.4\sqrt{2} - \frac{2}{5} - 2\ \left(\frac{2}{3}.2\sqrt{2} - \frac{2}{3}\right)$$

$$= \frac{8\sqrt{2}}{5} - \frac{2}{5} - \frac{8\sqrt{2}}{3} + \frac{4}{3}$$

Finalement $\boxed{K = \frac{-16\sqrt{2}+14}{15}}$

L = $\int_{-1}^{1} (1-x)\ \sqrt{1-x^2}\ dx$ On posera x = cos t

On pose x = cos t soit dx = - sin t dt et t = Arc os x

$$L = \int_{Ar\cos(-1)}^{Ar\cos(1)} (1-\cos t)\sqrt{1-\cos^2 t}\,(-\sin t)dt$$

$$= \int_{-p}^{0} (1-\cos t)\sqrt{\sin^2 t}\,(-\sin t)dt$$

$= \int_{-p}^{0} (1-\cos t)(-\sin t)(-\sin t)dt$ car sin t < 0 sur [- p ; p] donc $\sqrt{\sin^2 t}$ = - sin t

$= \int_{-p}^{0} (1-\cos t)(\sin^2 t)dt$

$= \int_{-p}^{0} (1-\cos t)(1-\cos^2 t)dt$ car sin²t + cos²t = 1

$= \int_{-p}^{0} (1-\cos^2 t - \cos t + \cos^3 t)dt$

$= \int_{-p}^{0} (1-(\frac{1+\cos 2t}{2}) - \cos t + \cos^3 t)dt$

$= \int_{-p}^{0} (1-(\frac{1+\cos 2t}{2}) - \cos t + (\frac{1}{4}\cos 3t + \frac{3}{4}\cos t)dt$ car $\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$

(voir chapitre complexes – linéarisation)

$= \int_{-p}^{0} (\frac{1}{2} - \frac{5}{8}\cos t - \frac{1}{2}\cos 2t + \frac{1}{4}\cos 3t)dt$

$= [\ \frac{t}{2} - \frac{5}{8}\sin t - \frac{1}{4}\sin 2t + \frac{1}{12}\sin 3t\ ]_{-p}^{0}$

= (0 – 0 -0 + 0) – ($\frac{-p}{2}$ - 0 – 0 + 0)

Finalement $\boxed{J = \frac{p}{2}}$

$M = \int_{0}^{1} \sqrt{\frac{1-x}{1+x}}\, dx$ On posera x = cos t

On pose x = cos t soit dx = - sin t dt et t = Arcos x

$M = \int_{Ar\cos(0)}^{Ar\cos(1)} \sqrt{\frac{1-\cos t}{1+\cos t}}(-\sin t)dt$

$= \int_{-\frac{p}{2}}^{0} \sqrt{\frac{(1-\cos t)(1-\cos t)}{(1+\cos t)(1-\cos t)}}(-\sin t)dt$

$= \int_{-\frac{p}{2}}^{0} \sqrt{\frac{(1-\cos t)^2}{1-\cos^2 t}}(-\sin t)dt$

$= \int_{-\frac{p}{2}}^{0} \sqrt{\frac{(1-\cos t)^2}{\sin^2 t}}(-\sin t)dt$

$= \int_{-\frac{p}{2}}^{0} \frac{\sqrt{(1-\cos t)^2}}{-\sin t}(-\sin t)dt$ car sur [- $\frac{p}{2}$;0] sin t < 0 donc $\sqrt{\sin^2 t} = -\sin t$

$= \int_{-\frac{p}{2}}^{0} \frac{1-\cos t}{-\sin t}(-\sin t)dt$ car sur $[-\frac{p}{2};0]$ $1 - \cos t > 0$ donc $\sqrt{1-\cos^2 t} = \cos t$

$= \int_{-\frac{p}{2}}^{0} (1-\cos t)dt$

$= [\, t - \sin t \,]_{-\frac{p}{2}}^{0}$

$= (0 - 0) - (-\frac{p}{2} - (-1))$

$= -(-\frac{p}{2} + 1)$

Finalement $\boxed{M = \frac{p}{2} - 1}$

$N = \int_{3}^{4} \frac{t+1}{(t-2)^2}\, dt$ On posera $x = t - 2$

On pose $x = t - 2$ soit $t = x + 2$ et $dx = dt$

$N = \int_{3-2}^{4-2} \frac{x+2+1}{x^2}dx$

$= \int_{1}^{2} \frac{x+3}{x^2}dx$

$= \int_{1}^{2} (\frac{x}{x^2} + \frac{3}{x^2})dx$

$= \int_{1}^{2} (\frac{1}{x} + \frac{3}{x^2})dx$

$= [\, \ln x - \frac{3}{x} \,]_{1}^{2}$

$= (\ln 2 - \frac{3}{2}) - (\ln 1 - 3)$

$= \ln 2 - \frac{3}{2} + 3$ Finalement $\boxed{N = \ln 2 + \frac{3}{2}}$

$O = \int_{0}^{1} \frac{dx}{(x^2+1)^2}$ On posera $x = \tan t$

On pose $x = \tan t$ soit $t = \text{Arctan } x$ et $dx = 1 + \tan^2 t = \frac{1}{\cos^2 t}dt$

$O = \int_{Arctan(0)}^{Arctan(1)} \frac{(1+\tan^2 t)dt}{(\tan^2 t + 1)^2}$

$$= \int_0^{\frac{p}{4}} \frac{dt}{1+\tan^2 t}$$

$$= \int_0^{\frac{p}{4}} \frac{dt}{\frac{1}{\cos^2 t}}$$

$$= \int_0^{\frac{p}{4}} \cos^2 t dt$$

$$= \int_0^{\frac{p}{4}} \frac{1+\cos 2t}{2} dt$$

$$= \frac{1}{2} [\ t + \frac{1}{2} \sin 2t\ ]_0^{\frac{p}{4}}$$

$$= \frac{1}{2} [(\frac{p}{4} + \frac{1}{2} \sin 2\frac{p}{4}) - (\ 0 + \frac{1}{2}.\sin 0)\ ]$$

$$= \frac{1}{2} (\frac{p}{4} + \frac{1}{2}.\ 1)$$ Finalement $$\boxed{O = \frac{p+2}{8}}$$

Exercice 2

1. $$I = \int_0^{\frac{p}{w}} t \sin(wt)\ dt$$

On pose $\begin{cases} u = t \\ v' = \sin(v\ t) \end{cases}$ On obtient $\begin{cases} u' = 1 \\ v = -\frac{1}{v}.\cos(v\ t) \end{cases}$

$$I = [-\frac{t}{v} \cos(v\ t)]_0^{\frac{p}{v}} - \int_0^{\frac{p}{v}} -\frac{1}{v} \cos(v\ t)\ dt$$

$$= \underline{\hspace{4cm}} + \frac{1}{v} \int_0^{\frac{p}{v}} \cos(v\ t)\ dt$$

$$= \underline{\hspace{4cm}} + \frac{1}{v} [\frac{1}{v} \sin(v\ t)]_0^{\frac{p}{v}}$$

$$= \underline{\hspace{4cm}} + \frac{1}{v^2} [\sin(v\ t)]_0^{\frac{p}{v}}$$

$$= -\frac{1}{v} (\frac{p}{v} \cos(v\ \frac{p}{v}) - 0 \cos 0) + \frac{1}{v^2} (\sin(v\ \frac{p}{v}) - \sin 0)\ )$$

$$= -\frac{p}{v^2} \cos p + \frac{1}{v^2} \sin p$$ Finalement $$\boxed{I = \frac{p}{v^2}}$$

2. $J = \int_0^{\frac{p}{w}} t(\cos(v\,t) + i\sin(v\,t))\,dt$

$J = \int_0^{\frac{p}{w}} t\cos(v\,t)\,dt + \underbrace{i\int_0^{\frac{p}{v}} t\sin(v\,t)\,dt}_{I}$

On a bien $\boxed{I = \mathrm{Im}(J)}$

3. $J = \int_0^{\frac{p}{w}} t\,e^{iwt}\,dt$

On pose $\begin{cases} u = t \\ v' = e^{ivt}\sin(v\,t) \end{cases}$ On obtient $\begin{cases} u' = 1 \\ v = \frac{1}{iv}e^{ivt} \end{cases}$

$J = [\,t\frac{1}{iv}e^{ivt}]_0^{\frac{p}{v}} - \int_0^{\frac{p}{v}} \frac{1}{iv}e^{ivt}\,dt$

$J = \_\_\_\_\_\_\_\_\_\_ - \frac{1}{iv}\int_0^{\frac{p}{v}} e^{ivt}\,dt$

$J = \_\_\_\_\_\_\_\_\_\_ - \frac{1}{iv}[\,\frac{1}{iv}e^{ivt}]_0^{\frac{p}{v}}$

$J = \_\_\_\_\_\_\_\_\_\_ - (\frac{1}{iv})^2[e^{ivt}]_0^{\frac{p}{v}}$

$J = \frac{1}{iv}[\,t\,e^{ivt}]_0^{\frac{p}{v}} + \frac{1}{v^2}[e^{ivt}]_0^{\frac{p}{v}}$

$J = \frac{1}{iv}(\frac{p}{v}e^{iv\frac{p}{v}} - 0\,.\,e^0) + \frac{1}{v^2}(e^{iv\frac{p}{v}} - e^0)$

$J = \frac{1}{iv}(\frac{p}{v}e^{ip}) + \frac{1}{v^2}(e^{ip} - 1)$

$J = \frac{p}{iv^2}e^{ip} + \frac{1}{v^2}(e^{ip} - 1)$ On utilise ensuite $e^{ip} = \cos p + i\sin p = -1 + i\,.\,0 = -1$

$J = \frac{p}{iv^2}(-1) + \frac{1}{v^2}(-1 - 1)$

$J = \frac{p}{iv^2}(-1) - \frac{2}{v^2}$

$J = \frac{pi}{i^2v^2}(-1) - \frac{2}{v^2}$

$J = \frac{pi}{(-1)v^2}(-1) - \frac{2}{v^2}$

$J = -\frac{2}{v^2} + i\,.\,\frac{p}{v^2}$ On retrouve bien $I = \mathrm{Im}(J) = \frac{p}{v^2}$ $\boxed{I = \frac{1}{v^2}}$